ON CYCLES AND COVERINGS ASSOCIATED TO A KNOT

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Abstract. Let $\mathcal{K}$ be a knot, G be the knot group, K be its commutator subgroup, and x be a distinguished meridian. Let Σ be a finite abelian group. The dynamical system introduced by D. Silver and S. Williams in [S],[SW1] consisting of the set $\mathrm{Hom}(K,\Sigma)$ of all representations $\rho : K \to \Sigma$ endowed with the weak topology, together with the homeomorhpism

$$\sigma_x : \mathrm{Hom}(K,\Sigma) \longrightarrow \mathrm{Hom}(K,\Sigma);\ \sigma_x\rho(a) = \rho(xax^{-1})\ \forall a \in K, \rho \in \mathrm{Hom}(K,\Sigma)$$

is finite, i.e. it consists of several cycles. In [L] we found the lengths of these cycles for $\Sigma = \mathbb{Z}/p$, p is prime, in terms of the roots of the Alexander polynomial of the knot, $\bmod p$. In this paper we generalize this result to a general abelian group Σ. This gives a complete classification of depth 2 solvable coverings over $S^3\backslash\mathcal{K}$.

Contents

1. Introduction

Let $\mathcal{K}$ be a knot, X be the knot complement in S^3 , $X = S^3 \setminus \mathcal{K}$, X_∞ be the infinite cyclic cover of X, and X_d be the cyclic cover of X of degree d.

Let G be the knot group, K be its commutator subgroup, and Σ be a finite group. Let x be a distinguished meridian of the knot. The dynamical system introduced by D. Silver and S. Williams in [S] and [SW1] consisting of the set $\mathrm{Hom}(K,\Sigma)$ of all representations $\rho : K \to \Sigma$ endowed with the weak topology, together with the homeomorhpism σ_x (the shift map):

$$\sigma_x : \mathrm{Hom}(K,\Sigma) \longrightarrow \mathrm{Hom}(K,\Sigma);\ \sigma_x\rho(a) = \rho(xax^{-1})\ \forall a \in K, \rho \in \mathrm{Hom}(K,\Sigma).$$

is a shift of finite type ([SW1]). Moreover, if Σ is abelian, this dynamical system is finite, i.e. it consists of several cycles ([SW2],[K]). In ([L]) we calculated the lengths of these cycles and their lcm (least common multiple) for $\Sigma = \mathbb{Z}/p$, p prime,

Date: January 11, 2013.

in terms of the roots of the Alexander polynomial of the knot, $\bmod p$. Our goal is to generalize these results to an arbitrary finite abelian group Σ. This gives a complete classification of solvable depth 2 coverings of $S^3 \setminus \mathcal{K}$. (By a solvable covering of depth n we mean a composition of n regular coverings $M_0 \to M_1 \to \ldots \to M_n$ with corresponding groups Γ_i, such that $\Gamma_0 \lhd \Gamma_1 \lhd \ldots \lhd \Gamma_n$ and Γ_{i+1}/Γ_i is abelian.)

Let $\Delta(t) = c_0 + c_1(t) + \ldots + c_n t^n$ be the Alexander polynomial of the knot $\mathcal{K}$, and $B - tA$ its Alexander matrix of size, say, $m \times m$, corresponding to the Wirtinger presentation. From [L] we know that

$$\mathrm{Hom}(K, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^n \quad \text{where } n = \deg(\Delta(t) \bmod p). \tag{1.1}$$

It turns out that the same result is true for a target group $\mathbb{Z}/p^r$:

$$\mathrm{Hom}(K, \mathbb{Z}/p^r) \cong (\mathbb{Z}/p^r)^n \quad \text{where } n = \deg(\Delta(t) \bmod p). \tag{1.2}$$

In section 2 we give a proof of (1.2) for two-bridge knots. In section 3 we prove a general result about solutions of the reccurence equation

$$Bx_j - Ax_{j+1} = 0, \tag{1.3}$$

where $x_i \in \mathcal{X}$, $\mathcal{X}$ and $\mathcal{Y}$ are finite modules, and A, $B : \mathcal{X} \to \mathcal{Y}$ are module homomorphisms. We then use this result in section 4 to prove (1.2) for an arbitrary knot. In section 5 we describe the set of periods and calculate their lcm for target group $\Sigma = \mathbb{Z}/p^r$, based on similar results for the target group $\mathbb{Z}/p$, obtained in [L]. We then generalize these results for any finite abelian group Σ.

In section 6 we describe the relation between the shift σ_x on $\mathrm{Hom}(K, \Sigma)$ and the pullback map τ^* corresponding to the meridian x, on the space of regular coverings over X_∞. In section 7 we construct a regular covering $p : N \to X_d$ with the group of deck transformations Σ, corresponding to a surjective homomorphism $\rho \in \mathrm{Hom}(K, \Sigma)$ with $\sigma_x^d \rho = \rho$, and prove that any regular covering of X_d with the group of deck transformations Σ can be obtained in this way. We conclude the paper by formulating our results in terms of $p-$adic representations of K and associated solenoids and flat principal bundles.

2. Case of a two-bridge knot

Let $\Delta(t)$ be the Alexander polynomial of a two-bridge knot $\mathcal{K}$ and n be the degree of $\Delta(t) \bmod p$. Since the Alexander polynomial is defined up to multiplication by $t^k, k \in \mathbb{Z}$, and has symmetric coeffitients, we can write

$$\Delta(t) = pd_k t^{-k} + \ldots + pd_1 t^{-1} + c_0 + c_1 t + \ldots + c_n t^n + pd_1 t^{n+1} + \ldots pd_k t^{n+k} t^{n+k},$$

where c_i, d_i are integers and $c_0 = c_n$ is not divisible by p. Similarly to the Theorem 9.1 in [L] we can prove that $\mathrm{Hom}(K, \mathbb{Z}/p^r)$ is isomorphic to the space of bi-infinite sequences $\{x_i\}_{i \in \mathbb{Z}}$, $x_i \in \mathbb{Z}/p^r$, satisfiing the following reccurence equation $\bmod p^r$:

$$\begin{gathered} pd_k x_{-k+j} + \ldots pd_1 x_{-1+j} + c_0 x_j + c_1 x_{j+1} + \ldots + c_n x_{n+j} + \\ + pd_1 x_{n+1+j} + \ldots + pd_k x_{n+k+j} = 0 \end{gathered} \tag{2.1}$$

From [L] we know that $\mathrm{Hom}(K, \mathbb{Z}/p) \cong (\mathbb{Z}/p)^n$ where $n = \deg(\Delta(t) \bmod p)$. The same is true for target groups $\mathbb{Z}/p^r$.

Theorem 2.1. $\mathrm{Hom}(K, \mathbb{Z}/p^r) \cong (\mathbb{Z}/p^r)^n$ *where* $n = \deg(\Delta(t) \bmod p)$.

Proof. We will prove that $x_0, x_1, \dots, x_{n-1} \in \mathbb{Z}/p^r$ uniquely determine the sequence $\{x_i\}_{i\in\mathbb{Z}}$, $x_i \in \mathbb{Z}/p^r$, satisfiing equation (2.1). The proof is by induction. For $r = 1$, given $x_0, x_1, \dots, x_{n-1} \in \mathbb{Z}/p$, x_n is uniquely determined $\bmod p$ by the equation

$$c_0x_0 + c_1x_1 + \ldots + c_nx_n = 0 \pmod p. \tag{2.2}$$

So, $x_0, x_1, \dots, x_{n-1} \bmod p$ uniquely determine the whole sequence $\{x_i\}_{i\in\mathbb{Z}} \pmod p$, satisfiing (2.1). This proves the base of induction.

Suppose the statement is true for r. Fix $x_0, x_1, \dots, x_{n-1} \mod p^{r+1}$ and let $\{x_i\}_{i\in\mathbb{Z}}$ be the sequence satisfiing equation:

$$pd_kx_{-k} + \ldots + pd_1x_{-1} + c_0x_0 + c_1x_1 + \ldots + c_nx_n + \ldots + pd_kx_{n+k} = 0 \mod p^r. \tag{2.3}$$

It is uniquely determined $\bmod p^r$, by induction assumption. But then all the terms of (2.3) except c_nx_n are determined $\bmod p^{r+1}$. So x_n and hence the whole sequence $\{x_i\}_{i\in\mathbb{Z}}$ is uniquely determined $\bmod p^{r+1}$ by $x_0, x_1, \dots, x_{n-1} \mod p^{r+1}$. □

3. Linear matrix reccurence equations

Theorem 3.1. *Let $\mathcal{X}, \mathcal{Y}$ be two finite modules of the same order, over the same ring R. Let $A, B : \mathcal{X} \longrightarrow \mathcal{Y}$ be modules homomorphisms such that* $\ker A \cap \ker B = 0$. *Consider the following reccurence equation:*

$$Bx_j - Ax_{j+1} = 0 \tag{3.1}$$

Then $\mathcal{X} = \mathcal{V} \oplus \mathcal{A} \oplus \mathcal{B}$, where $\mathcal{V} = \{v \in \mathcal{X}$: there exists a bi-infinite sequence $\ldots v_{-1}, v_0 = v, v_1, \ldots,$ satisfiing equation (3.1), }
$\mathcal{A} = \{a \in \mathcal{X}$: there exists an infinite sequence $\ldots, a_{-1}, a_0$ satisfiing (3.1) *and $a_{-i} = 0$ for sufficiently large i }.*
$\mathcal{B} = \{b \in \mathcal{X}$: there exists an infinite sequence $b_0 = b, b_1, b_2, \ldots,$ satisfying(3.1) *and $b_i = 0$ for sufficiently large i.}*

Proof. The proof is by induction in the order of $\mathcal{X}$ and $\mathcal{Y}$. Consider a diagram :

$$\begin{tikzcd}
 & \mathcal{X} \arrow[dl, "A"'] \arrow[d, "\pi_1"] \arrow[dr, "B"] & \\
\mathcal{Y} \arrow[dr, "\pi_2"'] & \mathcal{X}/\ker A \arrow[d, shift right, "\bar{A}"'] \arrow[d, shift left, "\bar{B}"] & \mathcal{Y} \arrow[dl, "\pi_2"] \\
 & \mathcal{Y}/B(\ker A) &
\end{tikzcd}$$

where by definition, π_1 and π_2 are factorization maps; $[x] = \pi_1(x)$; and

$$\bar{\mathcal{A}}([x]) = \pi_2 \circ A(x), \quad \bar{\mathcal{B}}([x]) = \pi_2 \circ B(x).$$

This diagram is not commutative, but its left- and right-hand triangles are commutative. Note that $\mathcal{X}/\ker A$ and $\mathcal{Y}/B(\ker A)$ are modules over R of the same order, since B is injective on $\ker A$.

Suppose that the statement of the theorem is true for $\mathcal{X}/\ker A$ and operators $\bar{A}$ and $\bar{B}$:

$$\mathcal{X}/\ker A = \bar{\mathcal{V}} \oplus \bar{\mathcal{A}} \oplus \bar{\mathcal{B}}, \tag{3.2}$$

where all the sequences in definition of $\bar{\mathcal{V}}$, $\bar{\mathcal{A}}$, $\bar{\mathcal{B}}$ satisfy the equation:

$$\bar{B}[x]_i - \bar{A}[x]_{i+1} = [0]. \tag{3.3}$$

Then we will prove that

$$\mathcal{X} = \mathcal{V} \oplus \mathcal{A} \oplus \mathcal{B}, \tag{3.4}$$

Take any $u \in \mathcal{X}$. By induction assumption $[u] = [v] + [a] + [b]$, where $[v] \in \bar{\mathcal{V}}$, $[a] \in \bar{\mathcal{A}}$, $[b] \in \bar{\mathcal{B}}$. We find lifts v, a, b of $[v]$, $[a]$, $[b]$ to $\mathcal{V}$, $\mathcal{A}$, $\mathcal{B}$ respectively. Let $\dots, [v_{-1}], [v_0] = [v], [v_1], \dots$ satisfy $\bar{B}[v_i] - \bar{A}[v_{i+1}] = [0]$, $i \in \mathbb{Z}$. Take any lift $\dots, y_{-1}, y_0, y_1, \dots$. Then $By_i - Ay_{i+1} = x_i \in B(\ker A)$. So $x_i = Bw_i$ for some $w_i \in \ker A$. Then

$$B(y_i - w_i) - A(y_{i+1} - w_{i+1}) = 0.$$

So $v_i = y_i - w_i$ satisfy (3.1) and $v = v_0 \in \mathcal{V}$ is a desired lift of $[v]$.

Similarly, for $[a] \in \bar{\mathcal{A}}$ there exists a sequence $\dots, [a]_{-1}, [a_0] = [a]$, satisfying(3.3) with $[a]_{-i} = [0]$ for $i \geq N$. As before, we can find a lift $\{a_{-i}\}_{i \geq 0}$, satisfying $Ba_{-i} - Aa_{-(i-1)} = 0$. Note that $a_{-i} \in \ker A$ for $i \geq N$. We have

$$B \cdot 0 = Aa_{-N}.$$

But then the sequence $\dots, 0, 0, a_{-N}, a_{-(N-1)}, \dots, a_0$ also satisfies (3.1), so $a = a_0 \in \mathcal{A}$ is a desired lifting.

We repeat the same argument to prove that $[b]$ has a lift $b \in \mathcal{B}$. If $\{[b_i]\}_{i \geq 0}$ satisfies (3.3) and $[b_i] = 0$ for $i \geq N$, we find a lift $\{b_i\}_{i \geq 0}$ satisfying(3.1). Since $b_i \in \ker A$ for $i \geq N$, and $Bb_i - Ab_{i+1} = 0$, we have also $b_i \in \ker B$ for $i \geq N-1$, hence $b_i = 0$ for $i \geq N-1$, since by assumption $\ker A \cap \ker B = 0$. So $b = b_0 \in \mathcal{B}$ is a desired lift. Since $\pi_1(u) = \pi_1(v + a + b)$, $u = v + a + b + \tilde{a}$, where $\tilde{a} \in \ker A$ and so $\tilde{a} \in \mathcal{A}$. The step of induction is done.

Since we can interchange the roles of A and B, it remains to prove the statement of the theorem in the case when A and B are monomorphisms and hence are isomorphisms, since $|\mathcal{X}| = |\mathcal{Y}|$. In this case any element $x \in \mathcal{X}$ has a bi-infinite continuation $x_i = (A^{-1}B)^i x$, satisfying(3.1). The theorem is proven. □

4. Main result for a general knot

In this section we prove that the Theorem 2.1 holds for any knot. Let $B - tA$ be the Alexander matrix of a general knot $\mathcal{K}$ arising from the Wirtinger presentation of the knot group G. Here A, B are $m \times m$ matrices with elements $0, \pm 1$.

Theorem 4.1. *Dynamical system* $(\mathrm{Hom}(K, \Sigma), \sigma_x)$ *is conjugate to the left shift in the space of bi-infinite sequences* $\{y_j\}_{j \in \mathbb{Z}}$, $y_j \in (\Sigma)^m$ *satisfyingreccurence equation*

$$By_j - Ay_{j+1} = 0. \tag{4.1}$$

For the target group $\mathbb{Z}/p$ this result is proven in [L], Theorem 4.2. For a general abelian group Σ the proof is identical .

We can apply theorem (3.1) for modules $(\mathbb{Z}/p^r)^m$ and linear operators $A, B : (\mathbb{Z}/p^r)^m \to (\mathbb{Z}/p^r)^m$ given by matrices A and B to get

$$(\mathbb{Z}/p^r)^m = \mathcal{V}_r \oplus \mathcal{A}_r \oplus \mathcal{B}_r, \tag{4.2}$$

where $\mathcal{V}_r = \{y \in (\mathbb{Z}/p^r)^m$: there exists a bi-infinite sequence $\dots, y_{-1}, y_0 = y, y_1, \dots,$ satisfying equation (4.1)$\}$,
$\mathcal{A}_r = \{a \in (\mathbb{Z}/p^r)^m$: there exists an infinite sequence $\dots, a_{-1}, a_0 = a$, satisfying (4.1) and $a_{-i} = 0$ for sufficiently large i $\}$,
$\mathcal{B}_r = \{b \in (\mathbb{Z}/p^r)^m$: there exists an infinite sequence $b = b_0, b_1, b_2, \dots,$ satisfying (4.1) and $b_i = 0$ for sufficiently large i $\}$.

We will use the uniquiness of continuatuion that follows from the finiteness of $\mathrm{Hom}(K,\Sigma)$ for a finite abelian group Σ (see Proposition 3.7 [SW2] and Theorem 1 (ii) [K]). If $\{x_i\}_{i\in\mathbb{Z}}$ and $\{y_i\}_{i\in\mathbb{Z}}$ satisfy (4.1), then $x_0 = y_0$ implies $x_i = y_i \ \forall\, i$. In particular, for $a \in \mathcal{A}_r,\ a \neq 0,$ there is no infinite continuation to the right, satisfying (4.1), and for $b \in \mathcal{B}_r,\ b \neq 0,$ there is no infinite continuation to the left, satisfying (4.1). (Otherwise we would have two bi-infinite sequences: $\ldots, 0, 0, \ldots, a_0, a_1, \ldots$ and $\ldots, 0, 0, \ldots$.) So $\mathrm{Hom}(K, \mathbb{Z}/p^r)$ being isomorphic to the space of be-infinite sequences satisfying (4.1), is isomorphic to $\mathcal{V}_r$.

Since the only decomposition of $(\mathbb{Z}/p^r)^m$ as a direct sum of three groups is

$$(\mathbb{Z}/p^r)^m \cong (\mathbb{Z}/p^r)^{n_r} \oplus (\mathbb{Z}/p^r)^{l_r} \oplus (\mathbb{Z}/p^r)^{m_r} \text{ with } n_r + l_r + m_r = m,$$

it follows from (4.2) that $\mathcal{V}_r \cong (\mathbb{Z}/p^r)^{n_r}$. Consider the projection:

$$\begin{array}{c} (\mathbb{Z}/p^{r+1})^m = \mathcal{V}_{r+1} \oplus \mathcal{A}_{r+1} \oplus \mathcal{B}_{r+1} \\ \Big\downarrow{\scriptstyle \pi} \\ (\mathbb{Z}/p^r)^m = \mathcal{V}_r \oplus \mathcal{A}_r \oplus \mathcal{B}_r \end{array}$$

Clearly $\pi(\mathcal{V}_{r+1}) \subset \mathcal{V}_r,\ \pi(\mathcal{A}_{r+1}) \subset \mathcal{A}_r,\ \pi(\mathcal{B}_{r+1}) \subset \mathcal{B}_r$. It follows that n_r is the same for all r. Since from Theorem 5.5 [L] it immediately follows that $n_1 = \deg(\Delta(t) \bmod p)$, we have proven the following theorem:

Theorem 4.2. *For any knot,* $\mathrm{Hom}(K, \mathbb{Z}/p^r) \cong (\mathbb{Z}/p^r)^n$, *where* $n = \deg(\Delta(t) \bmod p)$.

5. Least common multiple

Proposition 5.1. *The dynamical system* $(\mathrm{Hom}(K, \mathbb{Z}/p^r), \sigma_x)$ *is isomorphic to* $(\mathcal{V}_r, T_r)$, *where* $T_r = (A|\mathcal{V}_r)^{-1}(B|\mathcal{V}_r)$.

Proof. Restrictions $A|\mathcal{V}_k$ and $B|\mathcal{V}_k$ are isomorphisms, since $\ker A \in \mathcal{A}_k$ and $\ker B \in \mathcal{B}_k$. Also $A\mathcal{V}_k = B\mathcal{V}_k$ since every element $v \in \mathcal{V}_k$ has continuation to the right and to the left: there exist v_{-1} and v_1 such that $Bv_{-1} = Av,\ Bv = Av_1$. So $T_r : \mathcal{V}_r \to \mathcal{V}_r$ is well defined, and since T_r is conjugate to the left shift in the space of sequences satisfiing equation(1.3), the formula $T_r = (A|\mathcal{V}_r)^{-1}(B|\mathcal{V}_r)$ is obvious. □

In [L] we calculated the set of periods of orbits and their lcm for dynamical system $(\mathrm{Hom}(K,\Sigma), \sigma_x)$ with $\Sigma = \mathbb{Z}/p$ in terms of orders and multiplicities of the roots of $\Delta(t) \bmod p$. Now we find the lcm and the set of periods for $\Sigma = \mathbb{Z}/p^r$.

Theorem 5.2. *Let* $d_r =$ *lcm of periods of orbits of* $(\mathrm{Hom}(K, \mathbb{Z}/p^r), \sigma_x)$. *Then either* $d_i = d_1\ \forall i$, *or* $\exists\, s \geq 1$ *such that* $d_1 = \ldots = d_s$, *and* $d_{s+i} = d_1 p^i$.

Proof. The following diagram commutes:

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{\pi} & \mathcal{V}_{k+1} & \xrightarrow{\pi} & \mathcal{V}_k & \xrightarrow{\pi} & \ldots & \xrightarrow{\pi} & \mathcal{V}_1 \\ & & \Big\downarrow{\scriptstyle T_{k+1}} & & \Big\downarrow{\scriptstyle T_k} & & & & \Big\downarrow{\scriptstyle T_1} \\ \ldots & \xrightarrow{\pi} & \mathcal{V}_{k+1} & \xrightarrow{\pi} & \mathcal{V}_k & \xrightarrow{\pi} & \ldots & \xrightarrow{\pi} & \mathcal{V}_1 \end{array}$$

Let $\mathcal{V} = \varprojlim \mathcal{V}_k,\ \mathcal{V}_k \subset (\mathbb{Z}_p)^m$, where $\mathbb{Z}_p$ is the set of $p-$adic numbers, and $T : \mathcal{V} \to \mathcal{V}$, $T = \varprojlim T_k$. We will use the same notations for module homorphisms and their matrices in the standard basis. Let $E_r,\ E$ denote the identity isomorphisms of

$(\mathbb{Z}/p^r)^n$ and $(\mathbb{Z}_p)^n$ respectively. We have $T_1^{d_1} = E_1$, so either $T^{d_1} = E$, and then $T_r^{d_1} = E_r\ \forall r$, or $T^{d_1} = E + p^s A$ for some $s \in \mathbb{Z},\ s \geq 1$, and not all elements of matrix A are divisible by p. In the later case $T_i^{d_1} = E_i,\ i = 1, \ldots, s$. Since

$$T^{d_1 \cdot k} = (E + p^s A)^k = E + kp^s A + C_k^2 p^{2s} A^2 + \ldots + p^{s \cdot k} A^k,$$

we have $T^{d_1 p} = E + p^{s+1} A_1$, where not all elements of A_1 are divisible by p, and, by induction, $T^{d_1 p^i} = E + p^{s+i} A_i,\ \forall i \geq 1$, where not all elements of A_i are divisible by p. Then $T_{s+i}^{d_1 p^i} = E_{s+i}$ and the statement of the theorem follows. □

Proposition 5.3. *Let $Q_r \subset \mathbb{N}$ be the set of all periods of $(\mathrm{Hom}(K, \mathbb{Z}/p^r), \sigma_x)$. Then $Q_r \subset Q_{r+1}$.*

Proof. If $\{x_j\}_{j \in \mathbb{Z}},\ x_j \in \mathbb{Z}/p^r$ is a sequence satisfiing reccurence equation (4.1) $\bmod p^r$ with period d, then $\{px_j\}_{j \in \mathbb{Z}},\ px_j \in \mathbb{Z}/p^{r+1}$ satisfies (4.1) $\bmod p^{r+1}$ and has the same period. □

Now we turn to a general finite abelian group Σ, which is isomorphic to a direct sum of cyclic groups:

$$\Sigma = \bigoplus_{i \in I} \mathbb{Z}/p_i^{r_i},\ I \subset \mathbb{N}.$$

Then

$$\mathrm{Hom}(K, \Sigma) = \bigoplus_{i \in I} \mathrm{Hom}(K, \mathbb{Z}/p_i^{r_i}) = \bigoplus_{i \in I} (\mathbb{Z}/p_i^{r_i})^{n_i}, \quad \text{where } n_i = \deg(\Delta(t) \bmod p_i),$$

and the original dynamical system is the product of dynamical systems:

$$(\mathrm{Hom}(K, \Sigma), \sigma_x) = \bigoplus_{i \in I} (\mathrm{Hom}(K, \mathbb{Z}/p_i^{r_i}), \sigma_x).$$

Taking sums of orbits with different periods, we obtain the following proposition:

Proposition 5.4. *(i) Let d_i be lcm of periods of orbits of $(\mathrm{Hom}(K, \mathbb{Z}/p_i^{r_i}), \sigma_x)$. Then lcm of periods of orbits of $(\mathrm{Hom}(K, \Sigma), \sigma_x)$ is $lcm\{d_i, i \in I\}$.*
(ii) Let Q_i be the set of periods of orbits of $(\mathrm{Hom}(K, \mathbb{Z}/p_i^{r_i}), \sigma_x)$. Then the set of periods for $(\mathrm{Hom}(K, \Sigma), \sigma_x)$ is

$$Q = \{lcm\{q_i, i \in I\},\ q_i \in Q_i\}.$$

6. Pullback τ^* on the space of coverings over X_∞

Let $p_\infty : X_\infty \longrightarrow X$ be the infinite cyclic covering over the complement of the knot, and let $\tau : X_\infty \to X_\infty$ be the deck tansformation corresponding to the loop x. We will now give a geometric description of the transformation σ_x earlier defined algebraicly.

Let us remind the pullback construction. Let $P : E \to B$ and $f : Y \to B$ be two continuous maps. $\Gamma_P = \{(e, b) : e \in E, b \in B, P(e) = b\} \subset E \times B$ is the graph of P. We have $\mathrm{id} \times f : E \times Y \to E \times B$. Then, by definition, the pullback of P by f, $f^*(P) :\ (\mathrm{id} \times f)^{-1} \Gamma_P \to Y$ is the projection onto the second coordinate. We have $(\mathrm{id} \times f)^{-1} \Gamma_P = \{(e, y) :\ e \in E, y \in Y, P(e) = f(y)\}$. The projection of this set

onto the first coordinate, $\tilde{f}$, is the lift of f, since the following diagram commutes:

$$\begin{tikzcd}
(e,y) \arrow[r, "\tilde{f}"] \arrow[d, "f^*(P)"'] & e \arrow[d, "P"] \\
y \arrow[r, "f"] & f(y)=P(e)
\end{tikzcd}$$

Note that if P is a (regular) covering then so is $f^*(P)$.

Let $a \in X_\infty$, $p_\infty(a) = x(0)$ and let $p : (M, y) \to (X_\infty, a)$ be the covering corresponding to a group $\Gamma \subset \pi_1(X_\infty, a)$, so that $p_*(\pi_1(M, y)) = \Gamma$. Let $p' : (M', y') \to (X_\infty, \tau^{-1}a)$ be the pull back of p by τ. It is a covering corresponding to the group $\tau_*^{-1}\Gamma \subset \pi_1(X_\infty, \tau^{-1}a)$. Then $\tau : X_\infty \to X_\infty$ lifts to a homeomorphism $\hat{\tau} : M' \to M$ such that $p \circ \hat{\tau} = \tau \circ p'$.

$$\begin{tikzcd}
(M',y') \arrow[r, "\hat{\tau}"] \arrow[d, "p'"'] & (M,y) \arrow[d, "p"] \\
(X_\infty,\tau^{-1}a) \arrow[r, "\tau"] & (X_\infty,a)
\end{tikzcd}$$

Let $\tilde{x}$ be the lift of x to X_∞ connecting $\tau^{-1}a$ to a. If $\hat{x}$ is the lift of $\tilde{x}$ to M' beginning at y' and ending at y'', then $p' : (M', y'') \to (X_\infty, a)$ is the covering corresponding to the group $\tilde{x}^{-1}(\tau_*^{-1}\Gamma)\tilde{x} \subset \pi_1(X_\infty, a)$.

Let $\mathcal{C}$ denote the space of all coverings of X_∞ up to the usual equivalence. Let $\mathcal{G}$ be the space of conjugacy classes of subgroups of $\pi_1(X_\infty, a) \approx K$. There is one-to-one correspondance between $\mathcal{C}$ and $\mathcal{G}$. In what follows we will not distinguish notationally between a covering and its equivalence class, and between a subgroup and its conjugacy class.

The pullback transformation $\tau^* : \mathcal{C} \to \mathcal{C}$, corresponds to the map $\tilde{\gamma} :\ \mathcal{G} \to \mathcal{G}$, $\tilde{\gamma} :\ \Gamma \mapsto \tilde{x}^{-1}(\tau_*^{-1}\Gamma)\tilde{x} \subset \pi_1(X_\infty, a)$, $\forall\, \Gamma \subset \pi_1(X_\infty, a)$, which turns into the map γ acting on the subgroups of $K \subset \pi_1(X, x(0))$: $\gamma(\Gamma) = x^{-1}\Gamma\, x$, $\forall\, \Gamma \subset K$.

Regular coverings of X_∞ correspond to normal subgroups $\Gamma \subset K$, which in turn correspond to representations $\rho \in \mathrm{Hom}(K, \Sigma)$ such that $\ker \rho = \Gamma$, in various groups Σ . The corresponding map on the space $\mathrm{Hom}(K, \Sigma)$ is σ_x, where $\sigma_x\rho(\alpha) = \rho(x\alpha x^{-1})$. Indeed, if $\Gamma = \ker \rho$, then $x^{-1}\Gamma x = \ker \sigma_x\rho$. In summary we can say that *the shift σ_x in the space* $\mathrm{Hom}(K, \Sigma)$ *defined algebraicly corresponds to the pullback action of the deck transformation τ in the space of regular coverings over X_∞.*

7. Coverings of finite degree

Theorem 7.1. *There is one-to-one correspondence between the surjective elements $\rho \in \mathrm{Hom}(K, \Sigma)$ such that $\sigma_x^d\rho = \rho$ and regular coverings $p : N \to X_d$ with the group of deck transformations Σ.*

Proof. Let ρ satisfy the condition of the theorem. Take a covering $p_\rho : M \to X_\infty$ corresponding to $\ker \rho$. Since $\sigma_x^d\rho = \rho$, this covering coincides with its $d-$time pullback: $\tau^{*d}p_\rho = p_{\sigma_x^d\rho} = p_\rho$. We can lift τ^d to $\zeta : M \to M$ so that the following

diagram commutes:

$$\begin{tikzcd}
(M,y') \arrow[r, "\zeta"] \arrow[d, "p_\rho"'] & (M,y) \arrow[d, "p_\rho"] \\
(X_\infty, \tau^{-d}a) \arrow[r, "\tau^d"] \arrow[dr, "p_\infty"'] & (X_\infty, a) \arrow[d, "p_\infty"] \\
& (X, x(0))
\end{tikzcd}$$

If $\rho : K \to \Sigma$ is onto then $\Sigma \cong K/\ker\rho$ acts on M in the standard way: if $\alpha \in \pi_1(X_\infty, a)$ is a loop and $\tilde{\alpha}$ is its lift to M starting at y, it ends at $\rho(a)(y)$. Clearly the action of Σ commutes with ζ. So Σ acts on the space of orbits of ζ, $N = M/\zeta$. These orbits project onto orbits of τ^d. Since $X_\infty/\tau^d = X_d$, we obtained a regular covering $p : N \to X_d$.

Now we prove that any regular covering over X_d with the group of deck transformations Σ can be obtained in this way: namely, for any covering (that is convinient to denote by) $p_2 : N \to X_d$ with Σ as the group of deck transformations, $\exists \rho \in \mathrm{Hom}(K, \Sigma)$ such that $\sigma_x^d(\rho) = \rho$ and the covering $\varepsilon_2 : M \to X_\infty$ corresponding to the subgroup $\ker\rho$, such that $N = M/\zeta$, ζ being a lift of τ^d.
Consider a diagram

$$\begin{tikzcd}
& N \arrow[d, "p_2"] \\
X_\infty \arrow[r, "p_1"'] & X_d
\end{tikzcd}$$

where p_2 is a regular covering with a group of deck transformations Σ, and p_1 is an infinite cyclic covering with the generator τ^d. Let us consider the pullback of p_2 by p_1. Let $M \subset N \times X_\infty$, $M = \{(a,x) \mid p_2 a = p_1 x\}$. Then we have two covering maps ε_1 and ε_2, $\varepsilon_1(a,x) = a$, $\varepsilon_2(a,x) = x$, such that the following diagram commutes:

$$\begin{tikzcd}
M \arrow[r, "\varepsilon_1"] \arrow[d, "\varepsilon_2"'] & N \arrow[d, "p_2"] \\
X_\infty \arrow[r, "p_1"'] & X_d
\end{tikzcd}$$

For $y \in X_\infty$, $(a_1, y), (a_2, y), \ldots, (a_s, y)$ are all preimages of y under ε_2, where $a_1, a_2, \ldots, a_s$ are all preimages of $x = p_1(y)$ under p_2, and $(a, y_1), (a, y_2), \ldots,$ are all preimages of $a \in N$ under ε_1, where $y_1, y_2, \ldots$ are all preimages of $p_2(a)$ under p_1.

Since τ^d is a generator of the group of deck transformations of p_1, $\zeta = (\mathrm{id}, \tau^d)$ is a generator of the group of deck transformations of ε_1, while $\{(\sigma, \mathrm{id}) | \sigma \in \Sigma\} \cong \Sigma$ is the group of deck transformations of ε_2.

For any $\beta \in K$ let $\tilde{\beta}$ be its lift to M starting at $(y_0, \beta(0))$ and ending at $(y_1, \beta(0))$, where $y_0, y_1 \in N$. There exists a unique $\sigma \in \Sigma$ such that $\sigma y_0 = y_1$. Take $\rho(\beta) = \sigma$. It is easy to see that $\beta \in \ker\rho$ iff $x^d(\tau^d \circ \beta)x^{-d} \in \ker\rho$. So, $\ker\rho = \ker\sigma_x^d(\rho)$. Since we can think of ρ as the homomorphism $\rho : K \to K/\ker\rho \cong \Sigma$, we have $\sigma_x^d(\rho) = \rho$. $\square$

8. p-ADIC SOLENOIDS

The above results can be summarized in terms of solenoids fibered over manifolds X and X_∞.

Let us have a family of coverings $p_n : S_n \to B$, $n = 0, 1, 2 \dots$, over the same m-dimensional manifold B. We say that they form a *tower* if there is a family of coverings $g_n : S_n \to S_{n-1}$ such that $p_n = p_{n-1} \circ g_n$. In this case we can form the *inverse limit* $\mathcal{S} = \varprojlim S_n$ by taking the space of sequences $\bar{z} = \{z_n\}_{n=0}^{\infty}$, $z_n \in S_n$ such that $g_n(z_n) = z_{n-1}$. Endow $\mathcal{S}$ with the weak topology. It makes the natural projection $p_\infty : \mathcal{S} \to B$, $\bar{z} \mapsto z_0$, a locally trivial fibration with Cantor fibers (as long as $\deg p_n \to \infty$). Moreover, $\mathcal{S}$ has a "horizontal" structure of m-dimensional lamination. If it is minimal (i.e., if all the leaves are dense in $\mathcal{S}$), it is called a *solenoid* over B.

If all the coverings p_n are regular with the group of deck transformations Σ_n, then $\mathcal{S}$ is a flat *principal* Σ-*bundle* over B with $\Sigma = \varprojlim \Sigma_n$. This means that
(i) $p_\infty : \mathcal{S} \to B$ is a locally trivial fibration with fiber Σ: $\forall b \in B$, $\exists\, U \subset B$, $U \ni b$ and a homeomorphism ϕ_U such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi_U"] \arrow[dr, "p_\infty"'] & & U \times \Sigma \arrow[dl] \\ & U & \end{tikzcd}$$

(ii) If $U \cap V \neq \emptyset$ and $h_{U \cap V}$ is defined by commutative diagram

$$\begin{tikzcd} & p^{-1}(U \cap V) \arrow[dl, "\phi_U"'] \arrow[dr, "\phi_V"] & \\ (U \cap V) \times \Sigma \arrow[rr, "h_{U,V}"] & & (U \cap V) \times \Sigma \end{tikzcd}$$

then $\exists\, a = a_{U,V} \in \Sigma$, such that $h_{U,V}(b, \sigma) = (b, \sigma + a)$.
In this case Σ acts on $\mathcal{S}$ preserving fibers, so that for all $\alpha \in \Sigma$ the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U) \arrow[r, "\phi_U"] \arrow[d, "T_\alpha"'] & U \times \Sigma \arrow[d, "(b,\sigma)\mapsto(b,\sigma+\alpha)"] \\ p^{-1}(U) \arrow[r, "\phi_U"] & U \times \Sigma \end{tikzcd}$$

(we consider the case of an abelian Σ).

Given a principal flat Σ-bundle and a point $b \in B$, we can consider the monodromy action of $K = \pi_1(B, b)$ on the fiber $p_\infty^{-1}(b)$. Each element $\gamma \in K$ acts as a translation by some $\rho(\gamma) \in \Sigma$. (Let us cover the immage of γ by neighborhoods $U_0, U_1, \dots, U_n$ from the definition of flat principal Σ-bundle, such that $U_i \cap U_{i+1} \neq \emptyset$, $U_n = U_0$. The monodromy action of γ on $p^{-1}(b) \approx \Sigma$ is the translation by $\rho(\gamma) = \sum_{i=0}^{n-1} \alpha_{U_i, U_{i+1}}$). This action gives us a representation $\rho : K \longrightarrow \Sigma$.

Vice versa, given a representation $\rho : K \to \Sigma$, we can construct a flat principal Σ-bundle over B by taking the *suspension* of the K-action. The suspension space $\mathcal{S}$ is defined as the quotient of $\Sigma \times \tilde{B}$, where $\tilde{B}$ is the universal covering of B, by the diagonal action of K: $(\sigma, y) \sim (\sigma + \rho(\alpha), \alpha(y))$ $\forall \sigma \in \Sigma$, $y \in \tilde{B}$ and $\alpha(y)$ being the application of $\alpha \in K \cong \pi_1(B, b)$ to y. Indeed, it is easy to see that if we choose

a base point $y \in \pi^{-1}b \subset \tilde{B}$, then the elements of $p_\infty^{-1}b \subset \mathcal{S}$ can be "enumerated" by elements of Σ, and that conditions (i) and (ii) in the definition of a flat principal Σ-bundle are satisfied.

Thus, the space $\mathcal{C}(\Sigma)$ of principlal flat Σ-bundles over B (mod a natural equivalence) is identified with the space of representations $\rho : K \to \Sigma$.

In the case of $B = X_\infty$ and $\Sigma = \mathbb{Z}_p$, where $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^r$ is the group of p-adic numbers, the space $\mathcal{C}(\mathbb{Z}_p)$ of flat principal $\mathbb{Z}_p$-bundles (mod natural equivalence) is identified with the space of p-adic representations $\mathrm{Hom}(K, \mathbb{Z}_p)$. To the bundle

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] & \mathcal{S} \arrow[d, "p_\infty"'] \\ & X_\infty \end{tikzcd}$$

corresponding to a representation ρ, there are associated $\mathbb{Z}/p^r$-bundles

$$\begin{tikzcd} \mathbb{Z}/p^r \arrow[r] & S_r \arrow[d, "p_r"'] \\ & X_\infty \end{tikzcd}$$

corresponding to homorphisms $\rho_r : K \to \mathbb{Z}/p^r$, where ρ_r is the composition

$$K \xrightarrow{\rho} \mathbb{Z}_p \xrightarrow{\pi} \mathbb{Z}/p^r \ ,$$

π being the natural projection. Clearly, S_r form a tower of coverings and $\mathcal{S} = \varprojlim S_r$.

Note that S_r is connected iff $\rho_r : K \to \mathbb{Z}/p^r$ is onto. In the case when all ρ_r are onto, $\mathcal{S}$ is a solenoid over X_∞. If for some r, ρ_r is not onto, S_r is disconnected.

The pullback action of the deck transformation τ on $\mathcal{C}(\mathbb{Z}_p)$ corresponds to the σ_x-action in $\mathrm{Hom}(K, \mathbb{Z}_p)$.

The latter space is a finite dimensional $\mathbb{Z}_p$-module. Let us endow it with the sup-norm. Then any invertible operator $A : \mathrm{Hom}(K, \mathbb{Z}_p) \to \mathrm{Hom}(K, \mathbb{Z}_p)$ becomes an isometry. Since $\mathrm{Hom}(K, \mathbb{Z}_p)$ is compact, A is *almost periodic* in the sense that the cyclic operator group $\{A^n\}_{n\in\mathbb{Z}}$ is precompact. The closure of this group is called the *Bohr compactification* of A (see [Lyu]). Theorem 5.2 provides us with a description of this group for σ_x:

Theorem 8.1. *The Bohr compactification of the operator*

$$\sigma_x : \mathrm{Hom}(K, \mathbb{Z}_p) \to \mathrm{Hom}(K, \mathbb{Z}_p)$$

is the inverse limit of the cyclic groups $\mathbb{Z}/d_n$ where the d_n are the least common multiplies described by Theorem 5.2.

We can also consider solvable coverings over the knot complement X described in §7. Taking their inverse limits, we obtain various solenoids over X.

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